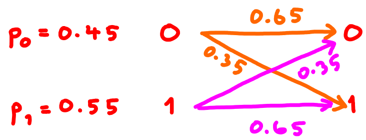
Q1 Optimal Decoder for Communication over BSC

Friday, September 12, 2014 11:48 AM

In this question, $p = 0.35$ and $p_0 = 0.45$

(a)



(b) First we find the Q matrix

$$Q = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.65 & 0.35 \\ 1 & 0.35 & 0.65 \end{array}$$

and the row vector $\underline{p}$

$$\underline{p} = \left[P[X=0] \quad P[X=1] \right]$$
$$= [p_0 \quad p_1] = [0.45 \quad 0.55]$$

To get matrix P, we simply scale each row of matrix Q by the corresponding $p(x)$:

$$Q = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.65 & 0.35 \\ 1 & 0.35 & 0.65 \end{array} \quad \begin{array}{c} \xrightarrow{\times 0.45} \\ \xrightarrow[\times 0.55]{} \end{array} \quad \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.2925 & 0.1575 \\ 1 & 0.1925 & 0.3575 \end{array} = P$$

(c) Recall that $\underline{q} = \underline{p}Q$.

For this question, $\underline{q} = [0.45 \quad 0.55] \begin{bmatrix} 0.65 & 0.35 \\ 0.35 & 0.65 \end{bmatrix} = [0.485 \quad 0.515]$

(d) This same table along with all the answers are shown in the lecture notes. For this question, we will simply use those results. Those who want more explanation on the derivation of results can take a look at the solution for the BAC one which is more general than the BSC case here.

$\hat{x}(y)$	$P[\hat{X}=0 \mid X=0]$	$P[\hat{X}=1 \mid X=1]$	$P(\mathcal{C})$	$P(\mathcal{E})$
y	$1-p = 0.65$	$1-p = 0.65$	$1-p = 0.65$	$p = 0.35$
$1-y$	$p = 0.35$	$p = 0.35$	$p = 0.35$	$1-p = 0.65$
1	0	1	$1-p_0 = 0.55$	$p_0 = 0.45$
0	1	0	$p_0 = 0.45$	$1-p_0 = 0.55$

(e) The MAP detector is optimal. So, we can simply compare $P(\mathcal{E})$ values.

From part (d), the optimal detector is $\hat{x}(y) = y$ because it has the lowest $P(\mathcal{E})$.

Therefore, $\hat{x}_{MAP}(y) = y$ and the corresponding $P(\mathcal{E}) = 0.35$

(f) From the Q matrix, we select the maximum in each column:

$x \backslash y$ 0 1 So, $y \mid \hat{x}_{ML}(y)$ Equivalently,

0 [0.65 0.35]

$$Q = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.65 & 0.35 \\ 1 & 0.35 & 0.65 \end{array}$$

So,

y	$\hat{x}_{ML}(y)$
0	0
1	1

corresponding x value of the selected value in each column.

Equivalently, $\hat{x}_{ML}(y) = y$.

From part (d), we know that the detector $\hat{x}(y) = y$ has $P(\mathcal{E}) = 0.35$.

Q2 Optimal Decoder for Communication over DMC

Friday, September 12, 2014 10:48 AM

$\underline{p} = [0.2, 0.4, 0.4]$

$$Q = \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ \hline 1 & 0.5 & 0.2 & 0.3 \\ 2 & 0.3 & 0.4 & 0.3 \\ 3 & 0.2 & 0.2 & 0.6 \end{array}$$

(a) We can get the P matrix by scaling each row of the Q matrix by the corresponding prior probability $p(x)$.

Therefore,

The convention for our class is that these numbers are ordered in the same way that they are specified in the support.

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.4} \\ \xrightarrow{\times 0.4} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.10 & 0.04 & 0.06 \\ 0.12 & 0.16 & 0.12 \\ 0.08 & 0.08 & 0.24 \end{bmatrix} \end{array} = P$$

(b) We can get $\underline{q}$ simply by applying the formula $\underline{q} = \underline{p}P$.

In this question, we have

$$\underline{q} = [0.2 \quad 0.4 \quad 0.4] \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} = [0.3 \quad 0.28 \quad 0.42]$$

(c) Naive Decoder : $\hat{x}_{Naive}(y) = y$

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.4} \\ \xrightarrow{\times 0.4} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.10 & 0.04 & 0.06 \\ 0.12 & 0.16 & 0.12 \\ 0.08 & 0.08 & 0.24 \end{bmatrix} \end{array} = P$$

For naive decoder, look at each column of P and select the element whose corresponding x value is the same as y in that column.

$P(\mathcal{C}) = 0.10 + 0.16 + 0.24 = 0.5$

$P(\mathcal{E}) = 1 - P(\mathcal{C}) = 1 - 0.5 = 0.5$

(d) DIY decoder : $\hat{x}_{DIY}(y) = 4 - y \Rightarrow$ Decoder table :

y	$\hat{x}(y)$
1	3
2	2
3	1

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.4} \\ \xrightarrow{\times 0.4} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.10 & 0.04 & 0.06 \\ 0.12 & 0.16 & 0.12 \\ 0.08 & 0.08 & 0.24 \end{bmatrix} \end{array} = P$$

For DIY decoder, look at each column of P and select the element whose corresponding x value is the same as $\hat{x}(y)$ in the decoder table.

$P(\mathcal{C}) = 0.08 + 0.16 + 0.06 = 0.30$

$P(\mathcal{E}) = 1 - 0.30 = 0.70$

(e) MAP Decoder:

$$Q = \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ \hline 1 & 0.5 & 0.2 & 0.3 \\ 2 & 0.3 & 0.4 & 0.3 \\ 3 & 0.2 & 0.2 & 0.6 \end{array} \quad \begin{array}{c} \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.4} \\ \xrightarrow{\times 0.4} \end{array} \quad \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ \hline 1 & 0.10 & 0.04 & 0.06 \\ 2 & 0.12 & 0.16 & 0.12 \\ 3 & 0.08 & 0.08 & 0.24 \end{array} = P$$

For MAP decoder, look at each column of P and select the element with max value.

The decoder table can be read from the selected values simply by finding the corresponding x value:

y	$\hat{x}_{MAP}(y)$
1	2
2	2
3	3

$P(\mathcal{C}) = 0.12 + 0.16 + 0.24 = 0.52$

$P(\mathcal{E}) = 1 - P(\mathcal{C}) = 1 - 0.52 = 0.48$

(f) ML Decoder:

For ML decoder, look at each column of Q and select the element with max value

$$Q = \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ \hline 1 & 0.5 & 0.2 & 0.3 \\ 2 & 0.3 & 0.4 & 0.3 \\ 3 & 0.2 & 0.2 & 0.6 \end{array} \quad \begin{array}{c} \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.4} \\ \xrightarrow{\times 0.4} \end{array} \quad \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ \hline 1 & 0.10 & 0.04 & 0.06 \\ 2 & 0.12 & 0.16 & 0.12 \\ 3 & 0.08 & 0.08 & 0.24 \end{array} = P$$

y	$\hat{x}_{ML}(y)$
1	1
2	2
3	3

The elements selected in the P matrix are simply those corresponding to positions selected in the Q matrix.

$P(\mathcal{C}) = 0.10 + 0.16 + 0.24 = 0.50$

$P(\mathcal{E}) = 1 - P(\mathcal{C}) = 1 - 0.5 = 0.5$

In fact, we know that $P(\mathcal{E})$ should be 0.5 without this calculation because $\hat{x}_{ML}(y)$ turns out to be the same as the naive decoder which we calculated $P(\mathcal{E}) = 0.5$.

Q3 MATLAB Simulation of the system in Q2

Saturday, September 13, 2014 11:44 PM

```
% MATLAB Script for Q3 of HW2 for ECS 452
% By Asst. Prof. Dr. Prapun Suksompong.

close all; clear all;
tic

load HW_DMC_Channel_Data

% Part
(a) ----------------------------------------------------------------
S_X = unique(x)
% Part
(b) ----------------------------------------------------------------
S_Y = unique(y)

%% Statistical Analysis
n = length(x);
% Part
(c) ----------------------------------------------------------------
% The probability values for the channel inputs
p_X_sim = hist(x,S_X)/n % Relative frequencies from the simulation
% Part
(d) ----------------------------------------------------------------
% The channel transition probabilities from the simulation
Q_sim = [];
for k = 1:length(S_X)
    I = find(x==S_X(k)); LI = length(I);
    rel_freq_Xk = LI/n;
    yc = y(I);
    cond_rel_freq = hist(yc,S_Y)/LI; Q_sim = [Q_sim; cond_rel_freq];
end
Q_sim                    % Relative frequencies from the simulation

% Part
(e) ----------------------------------------------------------------
% Part
(e.i) --------------------------------------------------------------
p_Y_sim = hist(y,S_Y)/n % Relative frequencies from the simulation
% Part
(e.ii) -------------------------------------------------------------
p_Y_sim2 = p_X_sim*Q_sim
%%
p_X = p_X_sim;
Q = Q_sim;
% Part
(f) ----------------------------------------------------------------
%% Naive Decoder
x_hat = y;
```

```
% Part
(f.i) ----------------------------------------------------------------
% Error Probability
PE_sim_Naive = 1-sum(x==x_hat)/n % Error probability from the simulation
% Part
(f.ii) ---------------------------------------------------------------
% Calculation of the theoretical error probability
PC = 0;
for k = 1:length(S_X)
    t = S_X(k);
    i = find(S_Y == t);
    if length(i) == 1
        PC = PC+ p_X(k)*Q(k,i);
    end
end
PE_theoretical_Naive = 1-PC
% Part
(g) ------------------------------------------------------------------
%% MAP Decoder
P = diag(p_X)*Q; % Weight the channel transition probability by the
                 % corresponding prior probability.
[V I] = max(P); % For I, the default MATLAB behavior is
                %that when there are multiple max, the
                % index of the first one is returned.
Decoder_Table_MAP = S_X(I) % The decoded values corresponding to the
                           % received Y
% Part
(g.i) ----------------------------------------------------------------
% Decode according to the decoder table
x_hat = y; % preallocation
for k = 1:length(S_Y)
    I = (y==S_Y(k));
    x_hat(I) = Decoder_Table_MAP(k);
end
PE_sim_MAP = 1-sum(x==x_hat)/n % Error probability from the simulation
% Part
(g.ii) ---------------------------------------------------------------
% Calculation of the theoretical error probability
Decoder_Table = Decoder_Table_MAP;
PC = 0;
for k = 1:length(S_X)
    I = (Decoder_Table == S_X(k));
    Q_row = Q(k,:);
    PC = PC+ p_X(k)*sum(Q_row(I));
end
PE_theoretical_MAP = 1-PC
% Part
(h) ------------------------------------------------------------------
%% ML Decoder
[V I] = max(Q); % For I, the default MATLAB behavior is
                %that when there are multiple max, the
                % index of the first one is returned.
Decoder_Table_ML = S_X(I) % The decoded values corresponding to the
```

```
                                % received Y
% Decode according to the decoder table
x_hat = y; % preallocation
for k = 1:length(S_Y)
    I = (y==S_Y(k));
    x_hat(I) = Decoder_Table_ML(k);
end
% Part
(h.i) --------------------------------------------------------------
PE_sim_ML = 1-sum(x==x_hat)/n % Error probability from the simulation
% Part
(h.ii) -------------------------------------------------------------
% Calculation of the theoretical error probability
Decoder_Table = Decoder_Table_ML;
PC = 0;
for k = 1:length(S_X)
    I = (Decoder_Table == S_X(k));
    Q_row = Q(k,:);
    PC = PC+ p_X(k)*sum(Q_row(I));
end
PE_theoretical_ML = 1-PC

toc
```

Results in the command window:

>> HW_DMC_Channel_Estimation_2

(a) S_X =

$\mathcal{X} = \{1 \quad 2 \quad 3\}$

(b) S_Y =

$\mathcal{Y} = \{1 \quad 2 \quad 3\}$

(c) p_X_sim =

$\underline{p} = [0.2004, \quad 0.4002, \quad 0.3994]$

(d) Q_sim =

$$Q = \begin{bmatrix} 0.4988 & 0.2012 & 0.3000 \\ 0.3002 & 0.4000 & 0.2998 \\ 0.2000 & 0.2003 & 0.5997 \end{bmatrix}$$

(e) p_Y_sim =

$\underline{q} = [0.3000, \quad 0.2804, \quad 0.4196]$

p_Y_sim2 =

$\underline{q} = [0.3000, \quad 0.2804, \quad 0.4196]$

(f) PE_sim_Naive =

$P(\mathcal{E}_{Naive}) = 0.5004$

PE_theoretical_Naive =

$P(\mathcal{E}_{Naive}) = 0.5004$

(g) Decoder_Table_MAP =

2 2 3 →

y	$\hat{x}_{MAP}(y)$
1	2
2	2
3	3

PE_sim_MAP =

$P(\mathcal{E}_{MAP}) = 0.4802$

PE_theoretical_MAP =

$P(\mathcal{E}_{MAP}) = 0.4802$

$P(\mathcal{E}_{MAP})$ = 0.4802

PE_theoretical_MAP =

$P(\mathcal{E}_{MAP})$ = 0.4802

(h) Decoder_Table_ML =

1 2 3

y	$\hat{x}_{ML}(y)$
1	1
2	2
3	3

PE_sim_ML =

$P(\mathcal{E}_{ML})$ = 0.5004

PE_theoretical_ML =

$P(\mathcal{E}_{ML})$ = 0.5004

Elapsed time is 0.410464 seconds.

Q4 Optimal Decoder for Communication over BAC

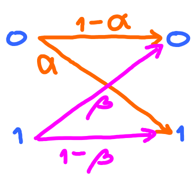
Friday, September 12, 2014 2:23 PM

For this question, we will first try to derive the answer in a general form. So, we will start with $Q(1|0) = \alpha$ and $Q(0|1) = \beta$.

After general expression is derived, we plug-in $\alpha = 0.35$, $\beta = 0.55$, $p_0 = 0.4$

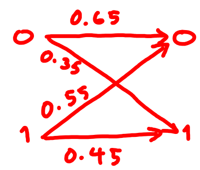
(a) $P[X=0] = p_0$

$P[X=1] = 1 - p_0$



$P[X=0] = 0.4$

$P[X=1] = 0.6$



(b)

First we find the Q matrix
$$Q = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 1-\alpha & \alpha \\ 1 & \beta & 1-\beta \end{array}.$$

To get matrix P, we simply scale each row of matrix Q by the corresponding $p(x)$:

$$Q = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 1-\alpha & \alpha \\ 1 & \beta & 1-\beta \end{array} \begin{array}{c} \xrightarrow{\times p_0} \\ \xrightarrow[\times 1-p_0]{} \end{array} \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & p_0(1-\alpha) & p_0\alpha \\ 1 & (1-p_0)\beta & (1-\beta)(1-p_0) \end{array} = P$$

For this question,

$$Q = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.65 & 0.35 \\ 1 & 0.55 & 0.45 \end{array} \begin{array}{c} \xrightarrow{\times 0.4} \\ \xrightarrow[\times 0.6]{} \end{array} \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.26 & 0.14 \\ 1 & 0.33 & 0.27 \end{array} = P$$

(c) Recall that $\underline{q} = \underline{p}Q$.

For this question, $\underline{p} = [p_0 \quad 1-p_0] = [0.4 \quad 0.6]$.

Therefore, $\underline{q} = [0.4 \quad 0.6] \begin{bmatrix} 0.26 & 0.14 \\ 0.33 & 0.27 \end{bmatrix} = [0.59 \quad 0.41]$

(d) First we consider the events $\hat{X} = 0$ and $\hat{X} = 1$.
We want to relate these events to the RV Y.

$\hat{x}(y)$	$[\hat{X} = 0]$	$[\hat{X} = 1]$
y	$[Y=0]$ ← $\hat{x}=0$ iff $Y=0$	$[Y=1]$ ← $\hat{x}=1$ iff $Y=1$
$1-y$	$[Y=1]$ ← $\hat{x}=0$ iff $Y=1$	$[Y=0]$ ← $\hat{x}=1$ iff $Y=0$
1	$\emptyset$ ← $\hat{x} \equiv 1$. So, it can not $= 0$	Ω ← $\hat{x} \equiv 1$. It always $= 1$. Regardless of the value of Y
0	Ω ← $\hat{x} \equiv 0$. It always $= 0$	$\emptyset$ ← $\hat{x} \equiv 0$. So, it can not $= 1$

With the above relationship, we can now find the conditional probabilities $P[\hat{X}=0|X=0]$ and $P[\hat{X}=1|X=1]$ by plugging in the corresponding Y-events above.

$\hat{x}(y)$	$P[\hat{X}=0 \mid X=0]$	$P[\hat{X}=1 \mid X=1]$

the corresponding Y-events above.

$\hat{x}(y)$	$P[\hat{X}=0 \mid X=0]$	$P[\hat{X}=1 \mid X=1]$
y	$P[Y=0 \mid X=0] = Q(0\mid 0) = 1-\alpha$	$P[Y=1 \mid X=1] = Q(1\mid 1) = 1-\beta$
$1-y$	$P[Y=1 \mid X=0] = Q(1\mid 0) = \alpha$	$P[Y=0 \mid X=1] = Q(0\mid 1) = \beta$
1	$P(\emptyset) = 0$	$P(\Omega) = 1$
0	$P(\Omega) = 1$	$P(\emptyset) = 0$

Finally, we have

Total probability theorem (Ω is partitioned by the events $[X=x]$)

$$P(\mathcal{C}) = P[\hat{X}=X] = \sum_{x} P[\hat{X}=X \mid X=x]\, P[X=x]$$

$$= \sum_{x} P[\hat{X}=x \mid X=x]\, p(x)$$

$$= P[\hat{X}=0 \mid X=0]\, \underbrace{p(0)}_{p_0} + P[\hat{X}=1 \mid X=1]\, \underbrace{p(1)}_{1-p_0}.$$

Therefore,

$\hat{x}(y)$	$P[\hat{X}=0 \mid X=0]$	$P[\hat{X}=1 \mid X=1]$	$P(\mathcal{C})$	$P(\mathcal{E})$
y	$1-\alpha$	$1-\beta$	$(1-\alpha)p_0 + (1-\beta)(1-p_0)$	$\alpha p_0 + \beta(1-p_0)$
$1-y$	α	β	$\alpha p_0 + \beta(1-p_0)$	$(1-\alpha)p_0 + (1-\beta)(1-p_0)$
1	0	1	$1-p_0$	p_0
0	1	0	p_0	$1-p_0$

Plugging in $\alpha = 0.35$, $\beta = 0.55$, $p_0 = 0.4$, we have

$\hat{x}(y)$	$P[\hat{X}=0 \mid X=0]$	$P[\hat{X}=1 \mid X=1]$	$P(\mathcal{C})$	$P(\mathcal{E})$
y	0.65	0.45	0.53	0.47
$1-y$	0.35	0.55	0.47	0.53
1	0	1	0.60	0.40
0	1	0	0.40	0.60

(e) The MAP detector is optimal. So, we can simply compare $P(\mathcal{E})$ values.

From part (d), the optimal detector is $\hat{x}(y) \equiv 1$ because it has the lowest $P(\mathcal{E})$.

Therefore, $\hat{x}_{MAP}(y) \equiv 1$ and the corresponding $P(\mathcal{E}) = 0.40$

Alternatively, we can also directly find the MAP detector following our recipe discussed in class:

$$Q = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.65 & 0.35 \\ 1 & 0.55 & 0.45 \end{array} \quad \begin{array}{c} \xrightarrow{\times 0.4} \\ \xrightarrow{\times 0.6} \end{array} \quad \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.26 & 0.14 \\ 1 & 0.33 & 0.27 \end{array} = P$$

$P(\mathcal{C}) = 0.33 + 0.27$
$= 0.60$
$P(\mathcal{E}) = 1 - P(\mathcal{C})$
$= 0.40$

For each column, select the max value and find the corresponding x-value.

y	$\hat{x}_{ML}(y)$
0	1
1	1

(f) From the Q matrix, we select the maximum in each column:

$$Q = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & & \\ 1 & & \end{array} \quad \begin{array}{cc} & \\ \left[\begin{array}{cc} 0.65 & 0.35 \\ 0.55 & 0.45 \end{array}\right] \end{array}$$

So,

y	$\hat{x}_{ML}(y)$
0	0
1	1

corresponding x value of the selected value in each column

Equivalently, $\hat{x}_{ML}(y) = y$.

From part (d), we know that the detector $\hat{x}(y) = y$ has $P(\mathcal{E}) = 0.47$.